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RAPID COMMUNICATIONS

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Drift-controlled anomalous diffusion: A solvable Gaussian model

Fabrizio Lillo and Rosario N. Mantegna
Istituto Nazionale per la Fisica della Materia, Unità di Palermo and Dipartimento di Fisica e Tecnologie Relative, Università di Palermo, Viale delle Scienze, I-90128 Palermo, Italy

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We introduce a Langevin equation characterized by a time-dependent drift. By assuming a temporal power-law dependence of the drift, we show that a great variety of behavior is observed in the dynamics of the variance of the process. In particular, diffusive, subdiffusive, superdiffusive, and stretched exponentially diffusive processes are described by this model for specific values of the two control parameters. The model is also investigated in the presence of an external harmonic potential. We prove that the relaxation to the stationary solution has a power-law behavior in time with an exponent controlled by one of the model parameters.

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Diffusive stochastic processes, i.e. stochastic processes $x(t)$ characterized by a linear growth in time of the variance $\langle x^2(t)\rangle \propto t$, are quite common in physical systems. However, deviations from a diffusive process are observed in several stochastic systems. Superdiffusive ($\langle x^2(t)\rangle \propto t^\nu$ with $\nu > 1$) and subdiffusive ($\langle x^2(t)\rangle \propto t^\nu$ with $\nu < 1$) random processes have been detected and investigated in physical and complex systems. A classical example of superdiffusive random process is Richardson's observation that two particles moving in a turbulent fluid which at time $t=0$ are originally placed very close to each other have a relative separation l at time t that follows the relation $\langle l^2(t)\rangle \propto t^3$ [1]. More recent examples include anomalous kinetics in chaotic dynamics due to flights and trapping [2,3], anomalous diffusion in aggregates of amphiphilic molecules [4], and anomalous diffusion in a two-dimensional rotating flow [5]. Subdiffusive stochastic processes have also been detected and investigated. Examples includes charge transport in amorphous semiconductors [6,7] and the dynamics of a bead in polymers [8]. Another class of stochastic processes which are not diffusive in a simple way is the one characterized by a variance with a stretched exponential time dependence. When a such process is Gaussian distributed the probability of return to the origin $P_0(t)$ is described by the Kohlrausch law $P_0(t) \propto \exp[-t^\nu]$ with $\nu < 1$. Similar behaviors are observed in glassy systems and in random walks in ultrametric spaces [9].

The modeling of some of the above discussed anomalous diffusing stochastic processes has been done by using a variety of approaches. To cite some examples, we recall that superdiffusive and subdiffusive processes have been modeled by writing down a generalized diffusion equation [1,10,11], by introducing Lévy walk models [12], by using a fractional Fokker-Planck equation approach [13], and by using *ad hoc* stochastic models such as the fractional Brownian motion [14].

In this Rapid Communication we introduce a class of Langevin equations capable of describing all the different anomalous regimes discussed above for Gaussian processes. Specifically, we study the properties of the class of Langevin equations

$$\dot{x} + \gamma(t)x = \Gamma(t), \qquad (1)$$

where $\gamma(t)$ is a function of time t and $\Gamma(t)$ is a Langevin force with zero mean and with a correlation function given by $\langle \Gamma(t_2)\Gamma(t_1)\rangle = D\,\delta(t_2 - t_1)$. Equation (1) describes an Ornstein-Uhlenbeck process [15] in the particular case of $\gamma(t) = \gamma$. This equation is linear and solvable. For the sake of simplicity we set the boundary condition of Eq. (1) at $t=0$. The formal solution of Eq. (1) is

$$x(t)=x(0)G(t)+G(t)\int_0^t \frac{\Gamma(s)}{G(s)}ds, \tag{2}$$

where $G(t)\equiv\exp[-\int_0^t \gamma(s)ds]$. By using this formal solution and all order correlation functions of $\Gamma(t)$ we obtain all central moments of $x(t)$. The first two central moments are given by

$$\langle x(t)\rangle=x(0)G(t)\equiv\mu(t),$$
$$\langle[x(t)-\mu(t)]^2\rangle=DG^2(t)\int_0^t \frac{1}{G^2(s)}ds\equiv\sigma^2(t). \tag{3}$$

The general relation between higher-order even central moments and the second central moment of the investigated processes is the one observed in a Gaussian process. Moreover, odd central moments are zero, hence we conclude that the stochastic processes described by Eq. (1) is Gaussian.

We now consider the two-time correlation functions of the process $x(t)$ and of its time derivative $\dot{x}(t)$. In the following we label the two times t_1 and t_2 of the correlation functions in such a way that $t_2\geqslant t_1$. By using the formal solution of Eq. (2) we determine the two-time correlation function for the random variable $x(t)$

$$\langle x(t_1)x(t_2)\rangle=\mu(t_1)\mu(t_2)+\frac{G(t_2)}{G(t_1)}\sigma^2(t_1). \tag{4}$$

In general, the correlation function $\langle x(t_1)x(t_2)\rangle$ is not a function of t_2-t_1 and therefore the process is usually nonstationary.

By starting from the correlation function of $x(t)$ and from the formal solution of the Langevin equation, we obtain the two-time correlation function of $\dot{x}(t)$ as

$$\langle\dot{x}(t_1)\dot{x}(t_2)\rangle=\mu_v(t_1)\mu_v(t_2)+D\delta(t_2-t_1)+\gamma(t_2)G(t_2)F(t_1), \tag{5}$$

where $\mu_v(t)=-\gamma(t)\mu(t)$ indicates the mean of the time derivative $\dot{x}(t)$ and $F(t_1)\equiv[\gamma(t_1)\sigma^2(t_1)-D]/G(t_1)$. The two-time correlation function of $\dot{x}(t)$ is the sum of a delta function and a smooth function.

The Fokker-Planck equation associated with the Langevin equation given in Eq. (1) is

$$\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial x}(\gamma(t)x\rho)+\frac{D}{2}\frac{\partial^2\rho}{\partial x^2}. \tag{6}$$

This Fokker-Planck equation is the same as the Smoluchowski equation of a Brownian particle moving in a harmonic oscillator with a time-dependent potential $U(x)\propto x^2\gamma(t)$. In our study we consider both positive and negative values of $\gamma(t)$. For positive values of $\gamma(t)$ the position $x=0$ is a stable equilibrium position, whereas in the opposite case $x=0$ is an unstable equilibrium position.

We calculate the two-time conditional probability density $P(x_2,t_2|x_1,t_1)$ as the Green function of the Fokker-Planck equation. In our formalism $t_1\leqslant t_2$. Our determination is made by working with the Fourier transform of $P(x_2,t_2|x_1,t_1)$ with respect to the x variable. The equation for the Fourier transform of $P(x_2,t_2|x_1,t_1)$ is a first-order partial differential equation, which can be solved by using characteristic curves. We obtain

$$P(x_2,t_2|x_1,t_1)=\frac{1}{\sqrt{2\pi s^2(t_2,t_1)}}\times\exp\left(-\frac{[x_2-m(t_2,t_1)x_1]^2}{2s^2(t_2,t_1)}\right), \tag{7}$$

where $m(t_2,t_1)=\exp[-\int_{t_1}^{t_2}\gamma(y)dy]$, and $s^2(t_2,t_1)=D\int_{t_1}^{t_2}\exp[-2\int_z^{t_2}\gamma(y)dy]dz$. Hence the transition probability of Eq. (7) is a Gaussian transition probability. Moreover, Eq. (7) satisfies the Chapman-Kolmogorov equation. In fact from a direct integration one can verify that $P(x_3,t_3|x_1,t_1)=\int P(x_3,t_3|x_2,t_2)P(x_2,t_2|x_1,t_1)dx_2$.

In the rest of this Rapid Communication we restrict our attention to the class of Langevin equations with a drift term that has temporal behavior of the form

$$\gamma(t)\sim a/t^\beta \tag{8}$$

for large time values. We study the stochastic process of Eq. (1) for different values of parameters a and β. Specifically, we focus on the asymptotic temporal evolution of the variance and of the two-time correlation function of $\dot{x}(t)$. We recall that for $a=0$, Eq. (1) describes a Wiener process with a variance increasing in a diffusive way, $\sigma^2(t)\sim t$, and a delta-correlated $\dot{x}(t)$. When $\beta=0$, Eq. (1) describes an Ornstein-Uhlenbeck process and one of two regimes is observed depending on the sign of a. When $a>0$ the stochastic process has a stationary Gaussian solution, whereas when $a<0$ there is no stationary state and the variance increases asymptotically in an exponential way: $\sigma(t)\sim\exp(2|a|t)$ [15,16]. The two-time correlation of the velocity decreases in an exponential way as $\exp[-|a|(t_2-t_1)]$.

The cases considered above are known. In addition to these cases, we observe a large variety of new behaviors controlled by the specific values of parameters a and β. By investigating the (β,a) set of parameters, we detect different anomalous behavior that we discuss below systematically by considering different regions of the β parameter.

(i) Region with $\beta>1$. The process $x(t)$ is diffusive and its variance increases linearly with time for any value of a. The two-time correlation function of $\dot{x}(t)$ can be obtained starting from Eq. (5). A direct calculation gives

$$\langle\dot{x}(t_1)\dot{x}(t_2)\rangle\sim\frac{a}{t_2^\beta}\exp\left(\frac{-at_2^{1-\beta}}{1-\beta}\right)F(t_1). \tag{9}$$

The process $\dot{x}(t)$ can be positively or negatively correlated depending on the sign of a. When $a>0$ $(a<0)$ the correlation is negative (positive). This property is valid for any value of β. By investigating the explicit form of Eq. (9) one observes that the correlation function decreases as a function of t_2 with a power-law dependence: $\langle\dot{x}(t_1)\dot{x}(t_2)\rangle\sim 1/t_2^\beta$.

TABLE I. Summary of the different diffusion regimes. The constant $C \equiv 2|a|/(1-\beta)$.

β	a	$\sigma^2(t)$	Description
β	$a=0$	t	Wiener (diffusive)
$\beta>1$	$a>0$	t	diffusive
$\beta>1$	$a<0$	t	diffusive
$\beta=1$	$a>-1/2$	t	diffusive
$\beta=1$	$a=-1/2$	$t \ln t$	log divergent
$\beta=1$	$a<-1/2$	$t^{2\|a\|}$	superdiffusive
$0<\beta<1$	$a>0$	t^{β}	subdiffusive
$0<\beta<1$	$a<0$	$\exp[Ct^{1-\beta}]$	less than exponentially diffusive
$\beta=0$	$a>0$	$1-\exp(-2at)$	Ornstein-Uhlenbeck
$\beta=0$	$a<0$	$\exp(2\|a\|t)$	exponentially diffusive
$\beta<0$	$a>0$	$1/t^{\|\beta\|}$	localized
$\beta<0$	$a<0$	$\exp[Ct^{1-\beta}]$	more than exponentially diffusive

(ii) Region with $\beta=1$. In this case we observe two regimes. When $a>-1/2$ the variance increases in a diffusive way: $\sigma^2(t)\sim t$. We find a different behavior when $a<-1/2$. In fact, by using Eq. (3) one can show that

$$\sigma^2(t)\sim t^{2|a|}. \tag{10}$$

Therefore, the particle performs a Gaussian superdiffusive random process. At the boundary value $a=-1/2$ the variance increases in a log-divergent way as $\sigma^2(t)\sim t\ln t$. The two-time correlation function of $\dot{x}(t)$ is determined starting from Eq. (5). An explicit calculation gives

$$\langle \dot{x}(t_1)\dot{x}(t_2)\rangle \sim \frac{a}{t_2^{1+a}}F(t_1). \tag{11}$$

The two-time correlation function of $\dot{x}(t)$ shows a power-law time dependence and the $\dot{x}(t)$ process is a strongly dependent random process [17]. We wish to point out that when $a=-1$ the diffusion of the $x(t)$ process is ballistic. This specific case has already been investigated by E. Nelson in the framework of stochastic mechanics. The Ito equation describing the stochastic process associated with the free evolution of a Gaussian quantum wave packet is [18]

$$dx(t)=\frac{t-c}{t^2+c^2}x\,dt+dw(t), \tag{12}$$

where $w(t)$ is a Wiener process and c is a constant. This stochastic equation describes the same random process of Eq. (1) for large values of t.

(iii) In the region $0<\beta<1$ we observe two regimes, which depend on the sign of a. When $a>0$ the variance increases as

$$\sigma^2(t)\sim t^{\beta}. \tag{13}$$

This behavior is the customary behavior observed in subdiffusive random process. The two-time correlation function of $\dot{x}(t)$ behaves asymptotically as

$$\langle \dot{x}(t_1)\dot{x}(t_2)\rangle \sim -\frac{aD}{t_2^{\beta}}\exp\left(-\frac{a}{1-\beta}(t_2^{1-\beta}-t_1^{1-\beta})\right). \tag{14}$$

If the time interval $\tau\equiv t_2-t_1$ is shorter than t_1 [19] the power-law term dominates in this equation and the stochastic process is power-law anticorrelated. For $\tau>>t_1$ the two-time correlation function decreases exponentially.

When $a<0$ the variance increases as a stretched exponential

$$\sigma^2(t)\sim \exp\left[\frac{2|a|}{1-\beta}t^{1-\beta}\right]. \tag{15}$$

Since the process is Gaussian, the probability of return to the origin follows the Kohlrausch law, $\rho(x(0),t)=1/\sqrt{2\pi}\sigma(t)\sim\exp[-t^{1-\beta}]$. This kind of anomalous diffusion has been observed in glasses and in random walks on an ultrametric space [9]. The two-time correlation function of $\dot{x}(t)$ increases with time as Eq. (14).

(iv) Region $\beta<0$. This region is essentially different from the previous ones because the absolute value of the drift term increases in time and eventually diverges. In this case we also observe two regimes depending on the sign of a. When $a<0$ the time evolution of the variance is formally the same as Eq. (15) of case (iii). In this region of β parameter the variance increases more than exponentially in time. When $a>0$ we find that the variance decreases with time with a power-law dependence $\sigma^2(t)\sim 1/t^{|\beta|}$. By using the Smoluchowski picture, we can interpret this behavior as the motion of a Brownian particle moving in a time-dependent potential, which leads to a localization of the particle in the point $x=0$. For both regimes the two-time correlation function of $\dot{x}(t)$ is given by Eq. (9).

We summarize the above-discussed variety of diffusive behavior of $\sigma^2(t)$ in Table I.

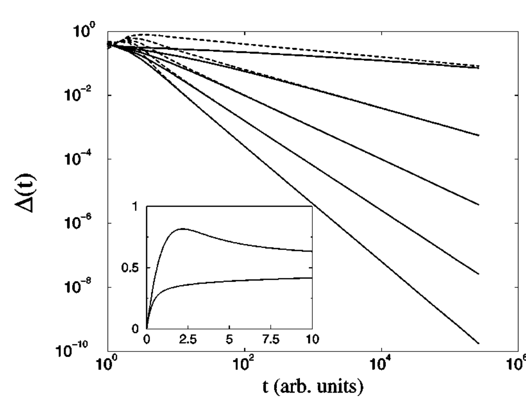

FIG. 1. Theoretical estimation of the normalized difference of the variance at time t from the stationary value σ_{st}^2, $\Delta(t)$ as a function of time. Different curves refer to different values of the control parameters a and β. The a values are $a=1$ (solid lines) and $a=-1$ (dashed lines). The parameter β assumes the values 0.2, 0.6, 1.0, 1.4, and 1.8 from top to bottom. In the inset we show a typical time evolution of $\sigma^2(t)$ obtained by setting $\beta=0.6$ and two different values of a, $a=1$ (bottom curve) and $a=-1$ (top curve). The other parameters are $\tau=1$, $k/M\eta=1$, and $D=1$

The results obtained above refer to $x(t)$ random processes which are not stationary. We now consider the problem of a process $x(t)$ whose dynamics is controlled by a modified version of Eq. (1) in which the effects of the presence of an ''external'' time-independent potential are taken into account. To this end we consider the specific case of an overdamped particle of mass M moving in a viscous medium in the presence of a potential having a time dependence of the kind described by Eq. (8) and a time-independent part. The equation of motion of such a system is

$$M\eta\dot{x}+g(t)x-F(x)=M\tilde{\Gamma}(t), \tag{16}$$

where η is the friction constant and $\tilde{\Gamma}(t)$ is the Langevin force with diffusion constant $2\eta k_B T/M$. This equation is formally equivalent to

$$\dot{x}+\gamma(t)x+V'(x)=\Gamma(t), \tag{17}$$

when $V'(x)=-F(x)/M\eta$, $\gamma(t)=g(t)/M\eta$, and $D=2k_B T/M\eta$. The prime in $V(x)$ indicates spatial derivative. It is worth pointing out that when $\gamma(t)$ goes to zero as t increases (as, for example, in the case $\gamma(t)\sim a/t^{\beta}$ with $\beta>0$), Eq. (17) might have a stationary solution. The presence of a stationary solution depends on the exact shape of $V(x)$.

To investigate in a concrete example the relaxation dynamics of the probability density function of $x(t)$ towards the stationary solution, we study Eq. (17) in the presence of an external harmonic potential, $V(x)=\frac{1}{2}kx^2$. In this case the process has a stationary state. A general solution of Eq. (17) is found by using the substitution $\gamma(t)\rightarrow\gamma(t)+k/M\eta$ in Eq. (2). In this case the variance of the process is equal to

$$\sigma^2(t)=De^{-2kt/M\eta}G^2(t)\int_0^t\frac{e^{2ks/M\eta}}{G^2(s)}ds. \tag{18}$$

The asymptotic stationary value of $\sigma^2(t)$ is $\sigma_{st}^2\equiv DM\eta/2k=k_B T/k$, which is independent of the parameters a and β. However, we observe a relaxation dynamics whose functional form is controlled by the values of a and β. To detect the different relaxation dynamics, we evaluate numerically the integral in Eq. (18) by setting $\gamma(t)=a/(\tau^{\beta}+t^{\beta})$. In Fig. 1 we show in a log-log plot the quantity $\Delta(t)\equiv|\sigma^2(t)-\sigma_{st}^2|/\sigma_{st}^2$ as a function of time. The quantity $\Delta(t)$ provides a measure of the distance of the system from the stationary behavior. In Fig. 1 we show that the quantity $\Delta(t)$ decreases following the power-law behavior $\Delta(t)\propto 1/t^{\beta}$ for large values of t and for all the investigated values of the parameters a and β. In particular, when $a>0$, $\sigma^2(t)-\sigma_{st}^2$ goes to zero as a negative value, whereas when $a<0$ the same quantity goes to zero taking positive values. In order to illustrate this result we show $\sigma^2(t)$ as a function of time when $\beta=0.6$ and $a=\pm 1$ in the inset of Fig. 1. For $0<\beta\leqslant 1$ (therefore including subdiffusive, superdiffusive, and stretched exponentially diffusing processes) it is not possible to define a characteristic time scale for the convergence of $\sigma^2(t)$ during the process of relaxation. This is due to the fact that the integral of $\int_{t_1}^{\infty}\Delta(t)/\Delta(t_1)dt=\infty$. Although a power-law behavior is still observed when $\beta>1$, it is worth pointing out that in this interval of β a typical time scale might be determined by considering the above-discussed integral which is finite in this region of β.

In conclusion, the Langevin equations (1) and (17) with the choice of Eq. (8) describe non-stationary and stationary random processes showing a wide class of (normal and anomalous) diffusion. When a stationary state exists, the relaxation dynamics to the stationary state has a power-law time dependence. The processes modeled by Eqs. (1) and (17) are characterized by a time-dependent drift term in the associated Fokker-Planck equation. Our model is complementary to Batchelor's description of anomalous diffusion obtained by assuming a time-dependent diffusion term [10]. Equations (1) and (17) can be used to model metastable systems in which one of the physical observables, such as the viscosity, is time dependent. They can also be used to develop simple and efficient algorithms generating realizations of random processes with controlled anomalous diffusion.

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